General Curvilinear Coordinates

Let the position of a point P in space is completely described by

$P(u_1, u_2, u_3)$, where

$u_1 = \text{Constant} = c_1$

$u_2 = \text{Constant} = c_2$

$u_3 = \text{Constant} = c_3$

u_1, u_2, u_3 are the three surfaces ~~where~~ which intersect at P

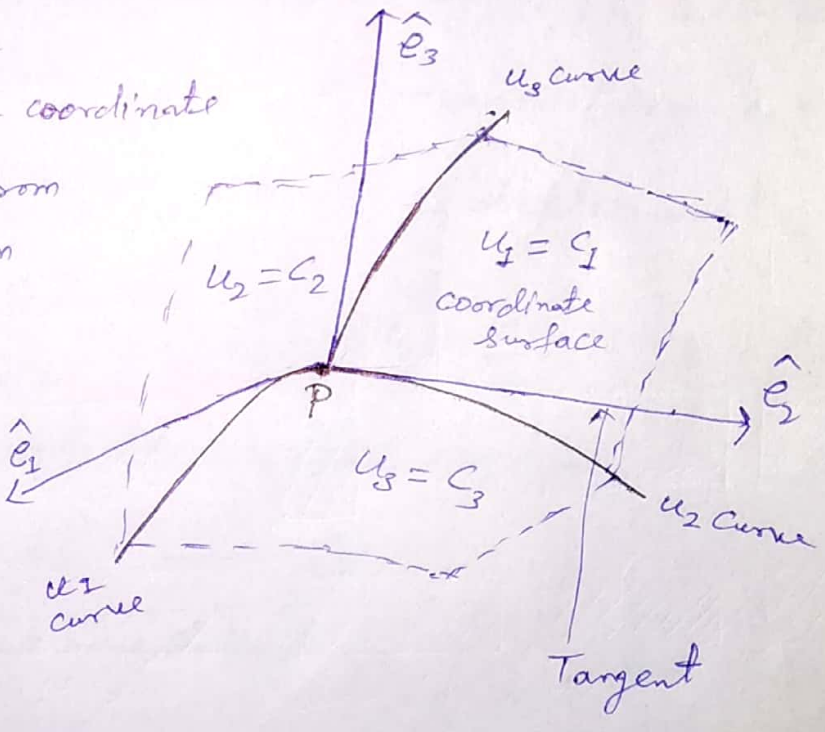
These surfaces are called coordinate surfaces

The three curves of intersection are called coordinate lines.

Tangents to the coordinate lines at P are called coordinate axes.

~~Note~~

* If the relative orientation of the coordinate surface changes from point to point, then (u_1, u_2, u_3) are called ~~general coordinate~~ ~~surf~~ general curvilinear coordinates.



* If the three surfaces $u_1 = c_1$, $u_2 = c_2$, $u_3 = c_3$ are mutually perpendicular, then (u_1, u_2, u_3) are called Orthogonal curvilinear Coordinates.

Let us consider that ~~rectangular~~ coordinates (x, y, z) of any point are expressed as functions of (u_1, u_2, u_3)

$$\left.\begin{aligned} x &= f_1(u_1, u_2, u_3) \\ y &= f_2(u_1, u_2, u_3) \\ z &= f_3(u_1, u_2, u_3) \end{aligned}\right\} \quad \text{——} \quad (1)$$

and above equations can be solved for u_1, u_2, u_3 in terms of x, y, z.

$$\left.\begin{aligned} u_1 &= F_1(x, y, z) \\ u_2 &= F_2(x, y, z) \\ u_3 &= F_3(x, y, z) \end{aligned}\right\} \quad \text{——} \quad (2)$$

Arc Length & volume Element: —

Next, we consider that $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ be the position vector of a point P. Then we can write (from Eqn (1))

$\vec{r} = \vec{r}(u_1, u_2, u_3)$; Element of displacement is given by

$$d\vec{r} = \frac{\partial \vec{r}}{\partial u_1} du_1 + \frac{\partial \vec{r}}{\partial u_2} du_2 + \frac{\partial \vec{r}}{\partial u_3} du_3 = \sum_i \frac{\partial \vec{r}}{\partial u_i} du_i$$

$$\cancel{= h_1 \hat{e}_1 du_1 + h_2 \hat{e}_2 du_2 + h_3 \hat{e}_3 du_3}$$

$$\equiv d\vec{s} \quad \text{——} \quad (3)$$

~~where we have defined $h_1 = \left|\frac{\partial \vec{r}}{\partial u_1}\right|$ —— (4)~~

~~$\hat{e}_1 = \frac{\partial \vec{r}}{\partial u_1} \Big/ \left|\frac{\partial \vec{r}}{\partial u_1}\right|$ —— (5)~~

~~h_1, h_2, h_3 are called scale factors~~

Further, square of the length of displacement $d\vec{s}$ is obtained as.

$$ds^2 = d\vec{r}\cdot d\vec{r} = \sum_{i=1}^{3}\sum_{j=1}^{3} \frac{\partial \vec{r}}{\partial u_i}\cdot\frac{\partial \vec{r}}{\partial u_j}\, du_i \cdot du_j$$

$$\text{or } ds^2 = \sum_{i=1}^{3}\sum_{j=1}^{3} \vec{a}_i\cdot\vec{a}_j\, du_i\cdot du_j$$

(we have replaced $\vec{a}_i = \frac{\partial \vec{r}}{\partial u_i}, a_j = \frac{\partial \vec{r}}{\partial u_j}$)

$$\boxed{ds^2 = \sum_{i=1}^{3}\sum_{j=1}^{3} g_{ij}\, du_i\, du_j} \quad \text{--- (4)} \quad \longleftarrow \text{general case}$$

where we have defined. $g_{ij} = \vec{a}_i\cdot\vec{a}_j$.

Note that g_{ij} is symmetric under exchange of i, j.

g_{ij} — called as metric coefficients.

Orthogonal curvilinear coordinates:

orthogonal condition is $\vec{a}_i\cdot\vec{a}_j = 0$, where $i \neq j$.

Now ds^2 is given by-

$$\boxed{ds^2 = g_{11}(du_1)^2 + g_{22}(du_2)^2 + g_{33}(du_3)^2} \text{ --- (5)}$$

↑ orthogonal case

It is to be noted that $du_2 = 0 = du_3$ when the element of length ds is along u_1. Thus, we write

$$ds_1 = \sqrt{g_{11}}\, du_1 = h_1\, du_1$$
$$ds_2 = \sqrt{g_{22}}\, du_2 = h_2\, du_2$$
$$ds_3 = \sqrt{g_{33}}\, du_3 = h_3\, du_3$$

Here, h_1, h_2, h_3 are the defined by $h_1 = \sqrt{g_{11}}$, $h_2 = \sqrt{g_{22}}$, $h_3 = \sqrt{g_{33}}$ and are called scale factors.

Scale factors – h_1, h_2, h_3.

$$h_1 = \left|\frac{\partial \vec{r}}{\partial u_1}\right|, \quad h_2 = \left|\frac{\partial \vec{r}}{\partial u_2}\right|, \quad h_3 = \left|\frac{\partial \vec{r}}{\partial u_3}\right|$$

Note :- For Cartesian coordinates, the metric coefficients $g_{11} = g_{22} = g_{33} = 1$..